

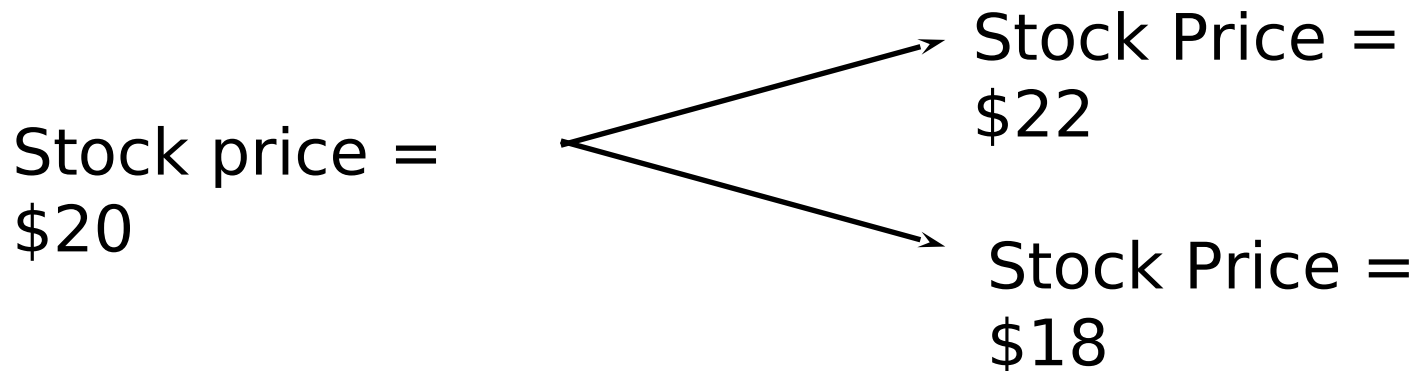


Binomial Trees

Chapter 11

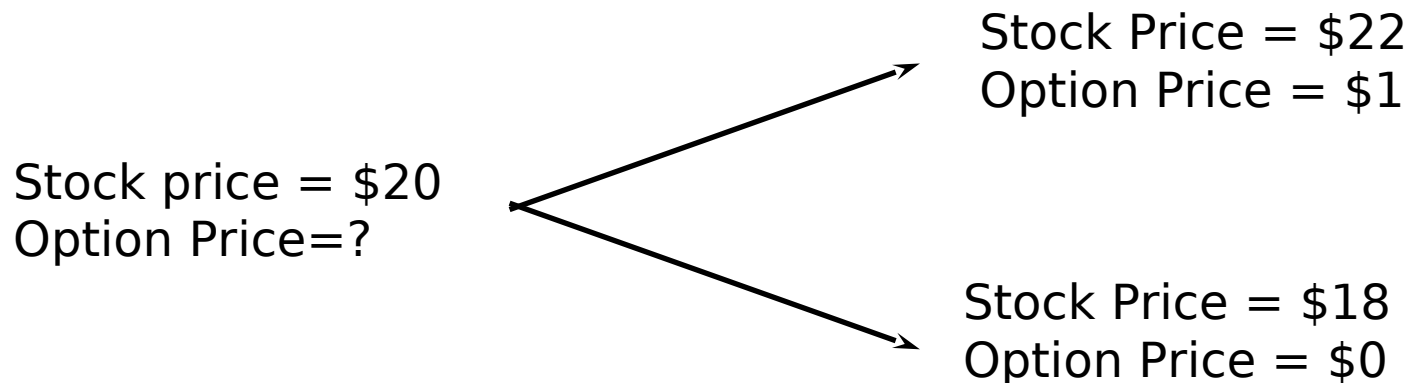
A Simple Binomial Model

- A stock price is currently \$20
- In 3 months it will be either \$22 or \$18



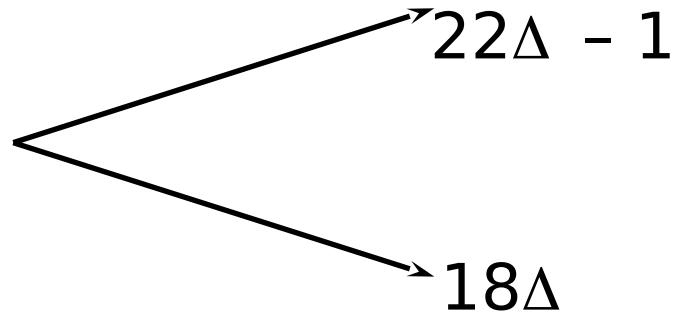
A Call Option (Figure 11.1, page 238)

A 3-month call option on the stock has a strike price of 21.



Setting Up a Riskless Portfolio

- Consider the Portfolio: long Δ shares short 1 call option
- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$



Valuing the Portfolio

(Risk-Free Rate is 12%)

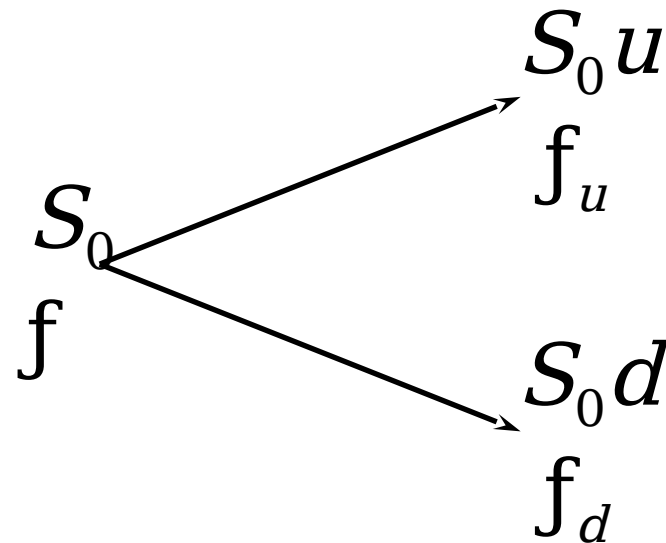
- The riskless portfolio is:
long 0.25 shares
short 1 call option
- The value of the portfolio in 3 months is $22 \times 0.25 - 1 = 4.50$
- The value of the portfolio today is $4.5e^{-0.12 \times 0.25} = 4.3670$

Valuing the Option

- The portfolio that is
 long 0.25 shares
 short 1 option
is worth 4.367
- The value of the shares is
 5.000 ($= 0.25 \times 20$)
- The value of the option is therefore
 0.633 ($= 5.000 - 4.367$)

Generalization (Figure 11.2, page 239)

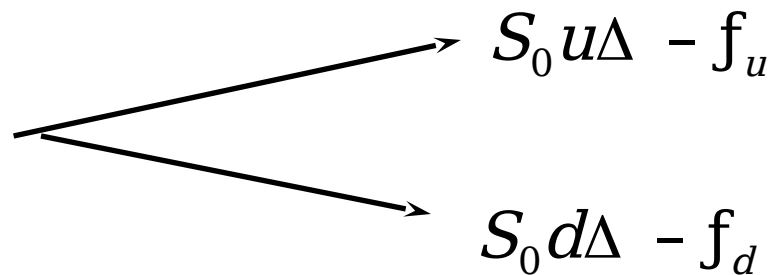
A derivative lasts for time T and is dependent on a stock



Generalization

(continued)

- Consider the portfolio that is long Δ shares and short 1 derivative



- The portfolio is riskless when $S_0 u \Delta - f_u = S_0 d \Delta - f_d$ or

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

Generalization

(continued)

- Value of the portfolio at time T is $S_0 u \Delta - f_u$
- Value of the portfolio today is $(S_0 u \Delta - f_u) e^{-rT}$
- Another expression for the portfolio value today is $S_0 \Delta - f$
- Hence

$$f = S_0 \Delta - (S_0 u \Delta - f_u) e^{-rT}$$

Generalization

(continued)

- Substituting for Δ we obtain

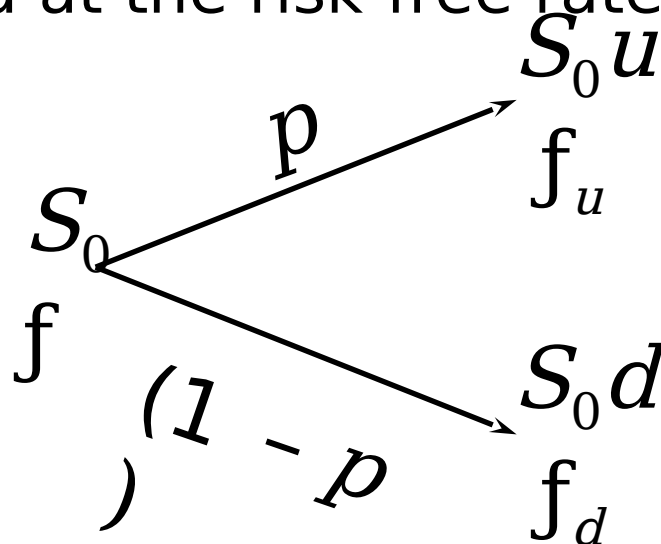
$$f = [pf_u + (1 - p)f_d]e^{-rT}$$

where

$$p = \frac{e^{rT} - d}{u - d}$$

p as a Probability

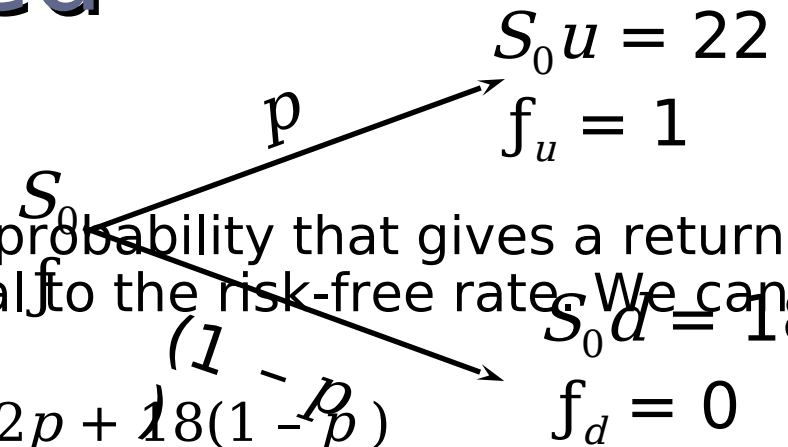
- It is natural to interpret p and $1-p$ as probabilities of up and down movements
- The value of a derivative is then its expected payoff in a risk-neutral world discounted at the risk-free rate



Risk-Neutral Valuation

- When the probability of an up and down movements are p and $1-p$ the expected stock price at time T is $S_0 e^{rT}$
- This shows that the stock price earns the risk-free rate
- Binomial trees illustrate the general result that to value a derivative we can assume that the expected return on the underlying asset is the risk-free rate and discount at the risk-free rate
- This is known as using risk-neutral valuation

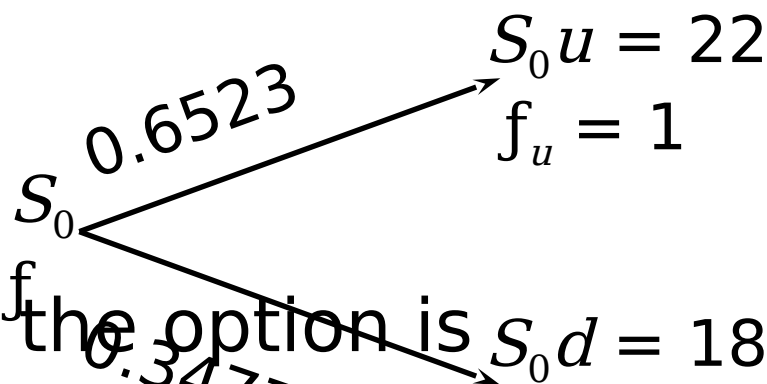
Original Example Revisited

- 
- Since p is the probability that gives a return on the stock equal to the risk-free rate. We can find it from

$$20e^{0.12 \times 0.25} = 22p + 18(1-p)$$
 which gives $p = 0.6523$
 - Alternatively, we can use the formula

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

Valuing the Option Using Risk-Neutral Valuation



The value of the option is

$$e^{-0.12 \times 0.25} (0.6523 \times 1 + 0.3477 \times 0)$$

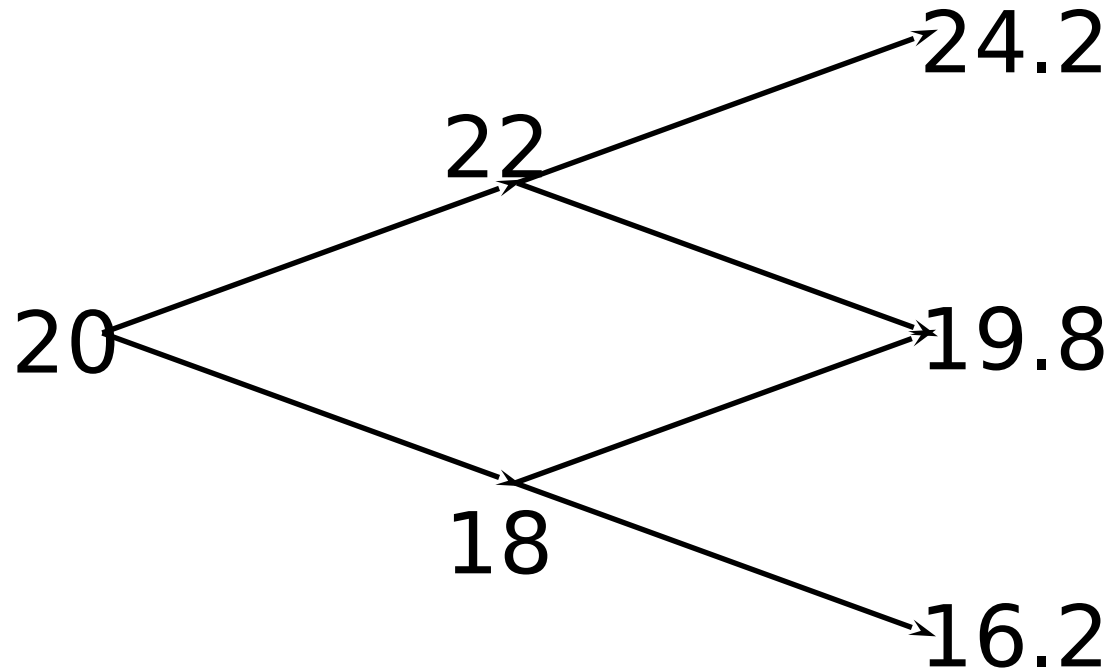
$$= 0.633$$

Irrelevance of Stock's Expected Return

- When we are valuing an option in terms of the price of the underlying asset, the probability of up and down movements in the real world are irrelevant
- This is an example of a more general result stating that the expected return on the underlying asset in the real world is irrelevant

A Two-Step Example

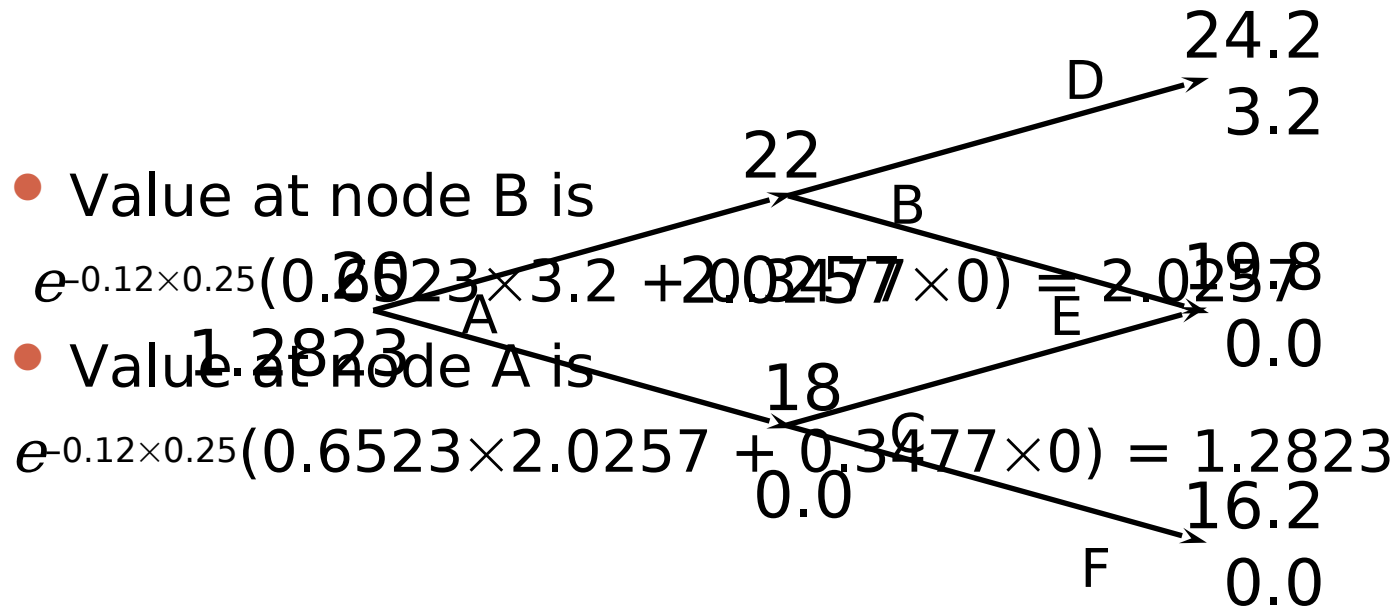
Figure 11.3, page 242



- Each time step is 3 months
- $K=21$, $r=12\%$

Valuing a Call Option

Figure 11.4, page 243

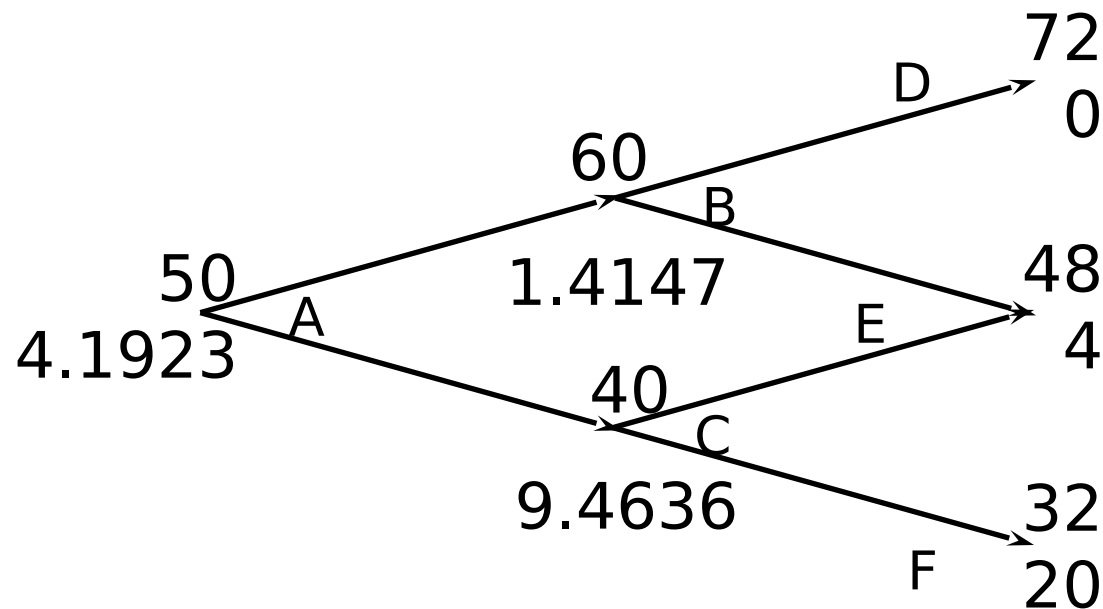


A Put Option Example

Figure 11.7, page 246

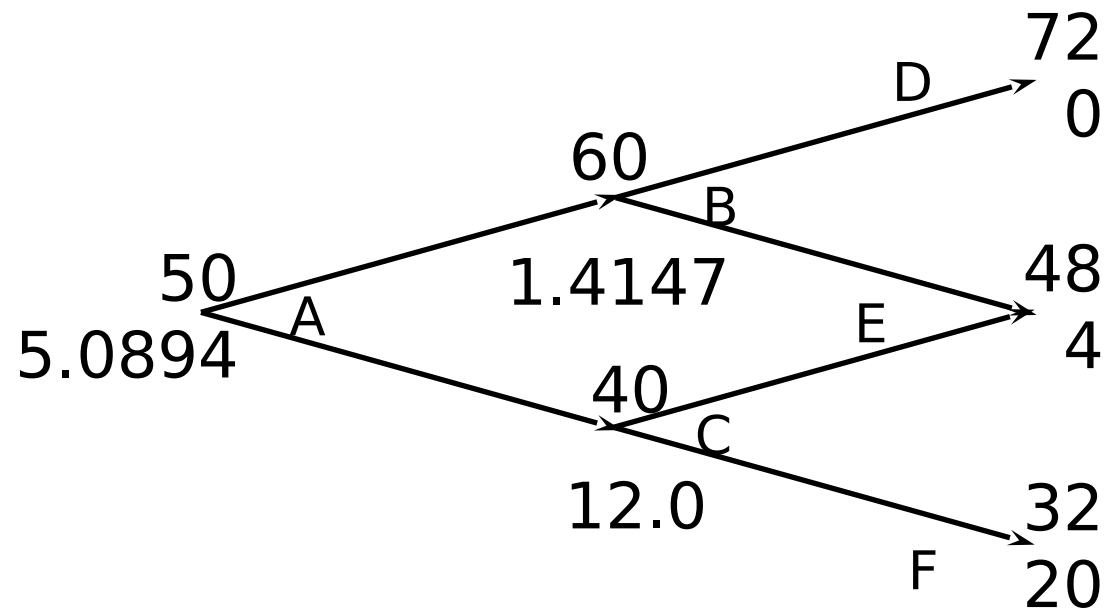
$K = 52$, time step = 1yr

$r = 5\%$



What Happens When an Option is American

(Figure 11.8, page 247)



Delta

- Delta (Δ) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock
- The value of Δ varies from node to node

Choosing u and d

One way of matching the volatility is to set

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

where σ is the volatility and Δt is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein

The Probability of an Up Move

$$p = \frac{a - d}{u - d}$$

- $a = e^{r\Delta t}$ for a nondividend-paying stock
- $a = e^{(r - q)\Delta t}$ for a stock index where q is the dividend yield on the index
- $a = e^{(r - r_f)\Delta t}$ for a currency where r_f is the foreign risk-free rate
- $a = 1$ for a futures contract